A VIEW OF MATHEMATICS

Alain CONNES
Institute des Hautes Etudes Scientifiques,France

Keywords: ESpace, projective geometry, non-Euclidean geometry, symmetry, Riemannian geometry, non-commutative geometry, motives, topos, positivity, cohomology, trace, index formula, Abelian category, quantum field theory, standard model, renormalization, cross ratio, line element.

Contents

1. The Unity of Mathematics
2. The concept of Space
2.1. Projective Geometry
2.2. The Angel of Geometry and the Devil of Algebra
2.3. Non-Euclidean Geometry
2.4. Symmetries
2.5. Line element and Riemannian geometry
2.6. Noncommutative Geometry
2.7. Grothendieck's Motives
2.8. Topos theory
3. Fundamental Tools
3.1. Positivity
3.2. Cohomology
3.3. Calculus
3.4. Trace and Index Formulas
3.5. Abelian Categories
3.6. Symmetries
4. The input from Quantum Field Theory
4.1. The Standard Model
4.2. Renormalization
4.3. Symmetries
Bibliography
Biographical Sketch

Summary

Mathematics is the backbone of modern science and a remarkably efficient source of new concepts and tools to understand the “reality" in which we participate. It plays a basic role in the great new theories of physics of the XXth century such as general relativity, and quantum mechanics.

The nature and inner workings of this mental activity are often misunderstood or simply ignored even among scientists of other disciplines. They usually only make use of rudimentary mathematical tools that were already known in the XIXth century and miss completely the strength and depth of the constant evolution of our mathematical concepts and tools.

The author was asked to write a general introduction to Mathematics which ended up as a rather personal point of view rather than producing the usual endless litany “X did this and Y did that". The evolution of the concept of “space" in mathematics serves as a unifying theme starting from some of its historical roots and going towards more recent developments in which the author has been more or less directly involved.

1. The Unity of Mathematics

It might be tempting at first to view mathematics as the union of separate parts such as Geometry, Algebra, Analysis, Number theory etc... where the first is dominated by the understanding of the concept of “space", the second by the art of manipulating “symbols", the next by the access to “infinity" and the “continuum" etc...

This however does not do justice to one of the most essential features of the mathematical world, namely that it is virtually impossible to isolate any of the above parts from the others without depriving them from their essence. In that way the corpus of mathematics does resemble a biological entity which can only survive as a whole and would perish if separated into disjoint pieces.

The first embryo of mental picture of the mathematical world one can start from is that of a network of bewildering complexity between basic concepts. These basic concepts themselves are quite simple and are the result of a long process of “distillation" in the alembic of the human thought.

Where a dictionary proceeds in a circular manner, defining a word by reference to another, the basic concepts of mathematics are infinitely closer to an “indecomposable element", a kind of “elementary particle" of thought with a minimal amount of ambiguity in their definition.

This is so for instance for the natural numbers where the number 3 stands for that quality which is common to all sets with three elements. That means sets which become empty exactly after we remove one of its elements, then remove another and then remove another. In that way it becomes independent of the symbol 3 which is just a useful device to encode the number.

Whereas the letters we use to encode numbers are dependent of the sociological and historical accidents that are behind the evolution of any language, the mathematical concept of number and even the specificity of a particular number such as 17 are totally independent of these accidents.

The “purity" of this simplest mathematical concept has been used by Hans Freudenthal to design a language for cosmic communication which he called “Lincos".

The scientific life of mathematicians can be pictured as a trip inside the geography of the “mathematical reality" which they unveil gradually in their own private mental frame.

It often begins by an act of rebellion with respect to the existing dogmatic description of that reality that one will find in existing books. The young “would be mathematicians" realize in their own mind that their perception of the mathematical world captures some features which do not quite fit with the existing dogma. This first act is often due in most cases to ignorance but it allows one to free oneself from the reverence to authority by relying on one’s intuition provided it is backed up by actual proofs. Once mathematicians get to really know, in an original and “personal" manner, a small part of the mathematical world, as esoteric as it can look at first (my starting point was localization of roots of polynomials), their trip can really start. It is of course vital all along not to break the “fil d’arianne" which allows to constantly keep a fresh eye on whatever one will encounter along the way, and also to go back to the source if one feels lost at times...

It is also vital to always keep moving. The risk otherwise is to confine oneself in a relatively small area of extreme technical specialization, thus shrinking one’s perception of the mathematical world and of its bewildering diversity.

The really *fundamental point* in that respect is that while so many mathematicians have been spending their entire scientific life exploring that world they all agree on its contours and on its connexity: whatever the origin of one’s itinerary, one day or another if one walks long enough, one is bound to reach a well known town *i.e.* for instance to meet elliptic functions, modular forms, zeta functions. “All roads lead to Rome" and the mathematical world is “connected".

In other words there is just “one" mathematical world, whose exploration is the task of all mathematicians and they are all in the same boat somehow.

Moreover exactly as the existence of the external material reality seems undeniable but is in fact only justified by the coherence and consensus of our perceptions, the existence of the mathematical reality stems from its coherence and from the consensus of the findings of mathematicians. The fact that proofs are a necessary ingredient of a mathematical theory implies a much more reliable form of “consensus" than in many other intellectual or scientific disciplines. It has so far been strong enough to avoid the formation of large gatherings of researchers around some “religious like" scientific dogma imposed with sociological imperialism.

Most mathematicians adopt a pragmatic attitude and see themselves as the explorers of this “mathematical world" whose existence they don’t have any wish to question, and whose structure they uncover by a mixture of intuition, not so foreign from “poetical desire"(as emphasized by the French poet Paul Valery), and of a great deal of rationality requiring intense periods of concentration.

Each generation builds a “mental picture" of its own understanding of this world and constructs more and more penetrating mental tools to explore previously hidden aspects of that reality.

Where things get really interesting is when unexpected bridges emerge between parts of the mathematical world that were previously believed to be very far remote from each other in the natural mental picture that a generation had elaborated. At that point one gets the feeling that a sudden wind has blown out the fog that was hiding parts of a beautiful landscape.

We shall see at the end of this chapter one recent instance of such a bridge. Before doing that we will take the concept of “space" as a guideline to take the reader through a guided tour leading to the edge of the actual evolution of this concept both in algebraic geometry and in noncommutative geometry. We shall also review some of the “fundamental" tools that are at our disposal nowadays such as “positivity", “cohomology", “calculus", “Abelian categories" and most of all “symmetries" which will be a recurrent theme in the three different parts of this text.

It is clearly impossible to give a “panorama" of the whole of mathematics in a reasonable amount of write up. But it is perfectly possible, by choosing a precise theme, to show the frontier of certain fundamental concepts which play a central role in mathematics and are still actively evolving.

The concept of “space" is sufficiently versatile to be an ideal theme to display this active evolution and we shall confront the mathematical concept of space with physics and more precisely with what Quantum Field Theory teaches us and try to explain several of the open questions and recent findings in this area.

2. The Concept of Space

The mental pictures of geometry are easy to create by exploiting the visual areas of the brain. It would be naive however to believe that the concept of “space" *i.e.* the stage where the geometrical shapes develop, is a straightforward one. In fact as we shall see below this concept of “space" is still undergoing a drastic evolution.

The Cartesian frame allows one to encode a point of the Euclidean plane (or space) by two (or three) real numbers $x^{\mu} \in \mathbb{R}$. This irruption of “numbers" in geometry appears at first as an act of violence undergone by geometry thought of as a synthetic mental construct.

This “act of violence" inaugurates the duality between geometry and algebra, between the eye of the geometer and the computations of the algebraists, which run in time contrasting with the immediate perception of the visual intuition.

Far from being a sterile opposition, this duality becomes extremely fecund when geometry and algebra become allies to explore unknown lands as in the new algebraic geometry of the second half of the twentieth century or as in noncommutative geometry, two existing frontiers for the notion of space.

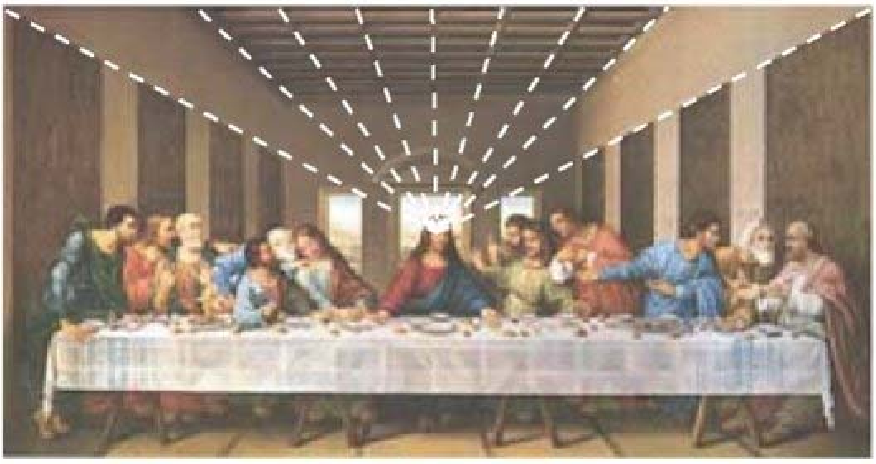

Figure 1: Perspective

2.1. Projective Geometry

Let us first briefly describe projective geometry a telling example of the above duality between geometry and algebra.

In the middle of the XVIIth century, G. Desargues, trying to give a mathematical foundation to the methods of perspective used by painters and architects founded real projective geometry. The real projective plane of Desargues is the set $P_2(\mathbb{R})$ of lines through the origin in three space $\mathbb{R}^3$. This adds to the usual points of the plane a “line at infinity" which gives a perfect formulation and support for the empirical techniques of perspective.

In fact Desargues’s theorem (Figure 2) can be viewed as the base for the axiomatization of projective geometry.

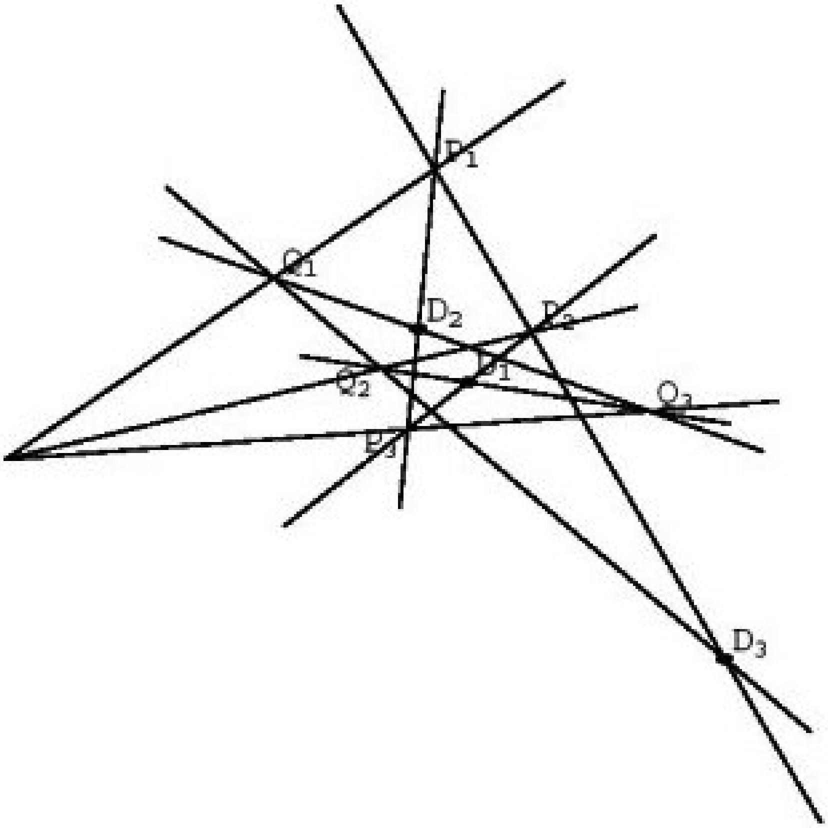

Figure 2: Desargues's Theorem : *Let* P_j *and* Q_j, $j \in \{1,2,3\}$ *be points such that the three lines* (P_j, Q_j) *meet. Then the three points* $D_j := (P_k, P_l) \cap (Q_k, Q_l)$ *are on the same line*.

This theorem is a consequence of the four extremely simple axioms which define projective geometry, but it requires for its proof that the dimension of the geometry be strictly larger than two.

These axioms express the properties of the relation "$P \in L$" *i.e.* the point P belongs to the line L, they are:

- Existence and uniqueness of the straight line containing two distinct points.
- Two lines defined by four points located on two meeting lines actually meet in one point.
- Every line contains at least three points.
- There exists a finite set of points that generate the whole geometry by iterating the operation passing from two points to all points of the line they span.

In dimension $n = 2$, Desargues's theorem is no longer a consequence of the above axioms and one has to add it to the above four axioms. The Desarguian geometries of dimension n are exactly the projective spaces $P_n(K)$ of a (not necessarily commutative) field K.

They are in this way in perfect duality with the key concept of algebra— that of field.

What is a field? It is a set of “numbers" that one can add, multiply and in which any non-zero element has an inverse so that all familiar rules are valid (except possibly the commutativity $xy = yx$ of the product). One basic example is given by the field $\mathbb{Q}$ of rational numbers but there are many others such as the field $\mathbb{F}_2$ with two elements or the field $\mathbb{C}$ of complex numbers. The field $\mathbb{H}$ of quaternions of Hamilton is a beautiful example of non-commutative field.

Complex projective geometry *i.e.* that of $P_n(\mathbb{C})$ took its definitive form in “La Géométrie" of Monge in 1795. The presence of complex points on the side of the real ones simplifies considerably the overall picture and gives a rare harmony to the general theory by the simplicity and generality of the results. For instance all circles of the plane pass through the “cyclic points" a pair of points (introduced by Poncelet) located on the line at infinity and having complex coordinates. Thus as two arbitrary conics any pair of circles actually meet in four points, a statement clearly false in the real plane.

The need for introducing and using complex numbers even to settle problems whose formulation is purely “real" had already appeared in the XVIth century for the resolution of the third degree equation. Indeed even when the three roots of such an equation are real the conceptual form of these roots in terms of radicals necessarily passes through complex numbers. (*cf.* Chapters 11 to 23 in Cardano’s book of 1545 *Ars magna sive de regulis algebraicis*).

2.2. The Angel of Geometry and the Devil of Algebra

The duality

$$\underline{\text{Geometry}} \quad | \quad \underline{\text{Algebra}} \tag{1}$$

already present in the above discussion of projective geometry allows us, when it is viewed as a mutual enhancement, to translate back and forth from geometry to algebra and obtain statements that would be hard to guess if one would stay confined to one of the two domains. This is best illustrated by a very simple example.

The geometric result, due to Frank Morley, deals with planar geometry and is one of the few results about the geometry of triangles that was apparently unknown to Greek mathematicians. You start with an arbitrary triangle ABC and trisect each angle, then you consider the intersection of consecutive trisectors, and obtain another triangle $\alpha\beta\gamma$ (Fig.3). Now Morley’s theorem, which he found around 1899, says that *whichever triangle ABC you start from, the triangle* $\alpha\beta\gamma$ *is always equilateral.*

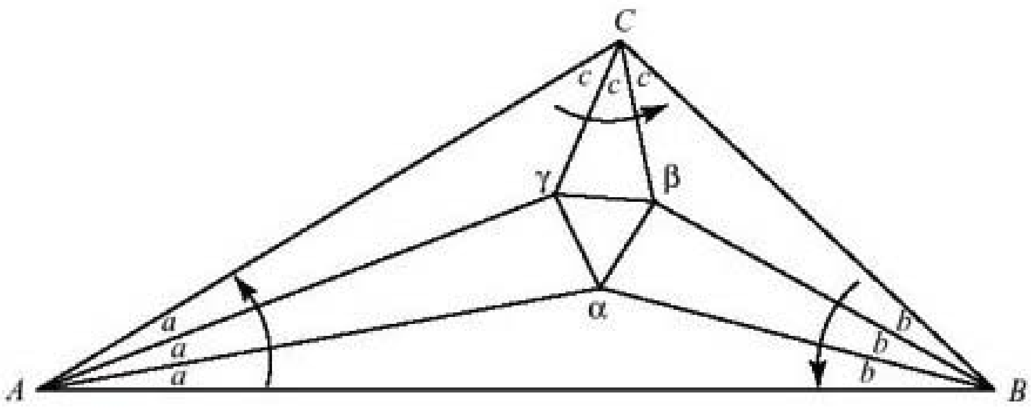

Figure 3: Morley's Theorem : *The triangle $\alpha\beta\gamma$ obtained from the intersection of consecutive trisectors of an arbitrary triangle ABC is always equilateral.*

Here now is an algebraic "transcription" of this result. We start with an arbitrary commutative field K and take three "affine" transformations of K. These are maps g from K to K of the form $g(x) = \lambda x + \mu$, where $\lambda \neq 0$. Given such a transformation the value of $\lambda \in K$ is unique and noted $\delta(g)$. For $g \in G$, $g(x) = \lambda x + \mu$ not a translation, *i.e.* $\lambda \neq 1$ one lets $\mathrm{fix}(g) = \alpha$ be the unique fixed point $g(\alpha) = \alpha$ of g. These maps form a group $G(K)$ (*cf.* Section 2.4) called the "affine group" and the algebraic counterpart of Morley's theorem reads as follows

Let $f, g, h \in G$ be such that fg, gh, hf and fgh are not *translations and let $j = \delta(fgh)$. The following two conditions are equivalent,*

a) $f^3 g^3 h^3 = 1$.

b) $j^3 = 1$ *and* $\alpha + j\beta + j^2\gamma = 0$ *where* $\alpha = \mathrm{fix}(fg)$, $\beta = \mathrm{fix}(gh)$, $\gamma = \mathrm{fix}(hf)$.

This is a sufficiently general statement now, involving an arbitrary field K and its proof is a simple "verification", which is a good test of the elementary skills in "algebra".

It remains to show how it implies Morley's result. But the fundamental property of "flatness" of Euclidean geometry, namely

$$a + b + c = \pi \qquad (2)$$

where a, b, c are the angles of a triangle (A, B, C) is best captured algebraically by the equality

$$F\,G\,H = 1$$

in the affine group $G(\mathbb{C})$ of the field $K = \mathbb{C}$ of complex numbers, where F is the rotation of center A and angle $2a$ and similarly for G and H. Thus if we let f be the rotation of center A and angle $2a/3$ and similarly for g and h we get the condition $f^3 g^3 h^3 = 1$.

The above equivalence thus shows that $\alpha + j\beta + j^2\gamma = 0$, where α, β, γ, are the fixed points of fg, gh et hf and where $j = \delta(fgh)$ is a non-trivial cubic root of unity. The relation $\alpha + j\beta + j^2\gamma = 0$ is a well-known characterization of equilateral triangles (it means $\frac{\alpha-\beta}{\gamma-\beta} = -j^2$, so that one passes from the vector $\overrightarrow{\beta\gamma}$ to $\overrightarrow{\beta\alpha}$ by a rotation of angle $\pi/3$).

Finally it is easy to check that the fixed point α, $f(g(\alpha)) = \alpha$ is the intersection of the trisectors from A and B closest to the side AB. Indeed the rotation g moves it to its symmetric relative to AB, and f puts it back in place. Thus we proved that the triangle (α, β, γ) is equilateral. In fact we also get for free 18 equilateral triangles obtained by picking other solutions of $f^3 = F$ etc...

This is typical of the power of the duality between on the one hand the visual perception (where the geometrical facts can be sort of obvious) and on the other hand the algebraic understanding. Then, provided one can write things in algebraic terms, one enhances their power and makes them applicable in totally different circumstances. For instance the above theorem holds for a finite field, it holds for instance for the field F_4 which has cubic roots of unity.... So somehow, passing from the geometrical intuition to the algebraic formulation allows one to increase the power of the original “obvious" fact, a bit like language can enhance the strength of perception, in using the “right words".

2.3. Non-Euclidean Geometry

The discovery of Non-Euclidean geometry at the beginning of the XIXth century frees the geometric concepts whose framework opens up in two different directions.

- The first opening is intimately related to the notion of symmetry and to the theory of Lie groups.
- The second is the birth of the geometry of curved spaces of Gauss and Riemann, which was to play a crucial role soon afterwards in the elaboration of general relativity by Einstein.

A particularly simple model of non-Euclidean geometry is the Klein model. The points of the geometry are those points of the plane which are located inside a fixed ellipse E (*cf.* Fig. 4). The lines of the geometry are the intersections of ordinary Euclidean lines with the inside of the ellipse.

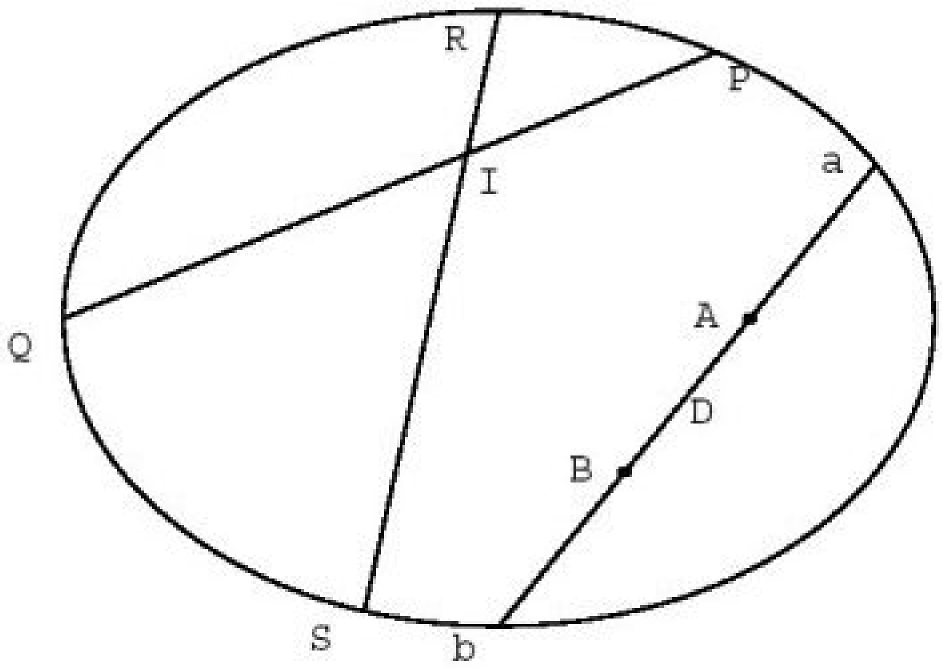

Figure 4: Klein model

The fifth postulate of Euclid on ‘flatness" *i.e.* on the sum of the angles of a triangle (2) can be reformulated as the uniqueness of the line parallel (*i.e.* not intersecting) to a given line D passing through a point $I \notin D$. In this form this postulate is thus obviously violated in the Klein model since through a point such as I pass several lines such as $L = PQ$ and $L' = RS$ which do not intersect D.

It is however not enough to give the points and the lines of the geometry to determine it in full. One needs in fact also to specify the relations of “congruence" between two segments (as well as between angles) AB and CD. The congruence of segments means that they have the same “length" and the latter is specified in the Klein model by

$$\text{length}(AB) = \log(\text{cross ratio}(A, B; b, a)) \tag{3}$$

where the cross-ratio of four points P_j on the same line with coordinates s_j is by definition

$$\text{cross ratio}(P_1, P_2; P_3, P_4) = \frac{(s_1 - s_3)(s_2 - s_4)}{(s_2 - s_3)(s_1 - s_4)} \tag{4}$$

Non-Euclidean geometry was discovered at the beginning of the XIXth century by Lobachevski and Bolyai, after many efforts by great mathematicians such as Legendre to show that the fifth Euclid’s axiom was unnecessary. Gauss discovered it independently and did not make his discovery public, but by developing the idea of “intrinsic curvature" he was already ways ahead anyway.

All of Euclid’s axioms are fulfilled by this geometry (These Euclid’s axioms are notably more complicated than those of projective geometry mentioned above.) except for the fifth one. It is striking to see, looking back, the fecundity of the question of the independence of the fifth axiom, a question which at first could have been hastily discarded as a kind of mental perversion in trying to eliminate one of the axioms in a long list that would not even look any shorter once done.

What time has shown is that far from just being an esoteric counterexample Non-Euclidean geometry is of a rare richness and fecundity. By breaking the traditional framework it generated two conceptual openings which we alluded to above and that will be discussed below, starting from the S. Lie approach.

2.4. Symmetries

One way to define the congruence of segments in the above Klein model, without referring to “length", *i.e.* to formula (3), is to use the natural symmetry group G of the geometry given by the projective transformations T of the plane that preserve the ellipse E. Then by definition, two segments AB and CD are congruent if and only if there exists such a transformation $T \in G$ with

$$T(A) = C\,,\ T(B) = D\,.$$

The set of these transformations forms a group *i.e.* one can compose such transformations and obtain another one, *i.e.* one has a “law of composition"

$$(S,T) \to S \circ T \in G, \quad \forall S,T \in G, \tag{5}$$

of elements of G in which multiple products are defined independently of the parenthesis, *i.e.*

$$(S \circ T) \circ U = S \circ (T \circ U)) \tag{6}$$

a condition known as “associativity", while the identity transformation id fulfills

$$S \circ \mathrm{id} = \mathrm{id} \circ S = S \tag{7}$$

and every element S of the group admits an inverse, uniquely determined by

$$S \circ S^{-1} = S^{-1} \circ S = \mathrm{id} \tag{8}$$

Group theory really took off with the work of Abel and Galois on the resolution of polynomial equations (*cf.* Section 3.6). In that case the groups involved are finite groups *i.e.* finite sets G endowed with a law of composition fulfilling the above axioms. Exactly as an integer can be prime *i.e.* fail to have non-trivial divisors a finite group can be “simple" *i.e.* fail to map surjectively to a smaller non-trivial group while respecting the composition rule.

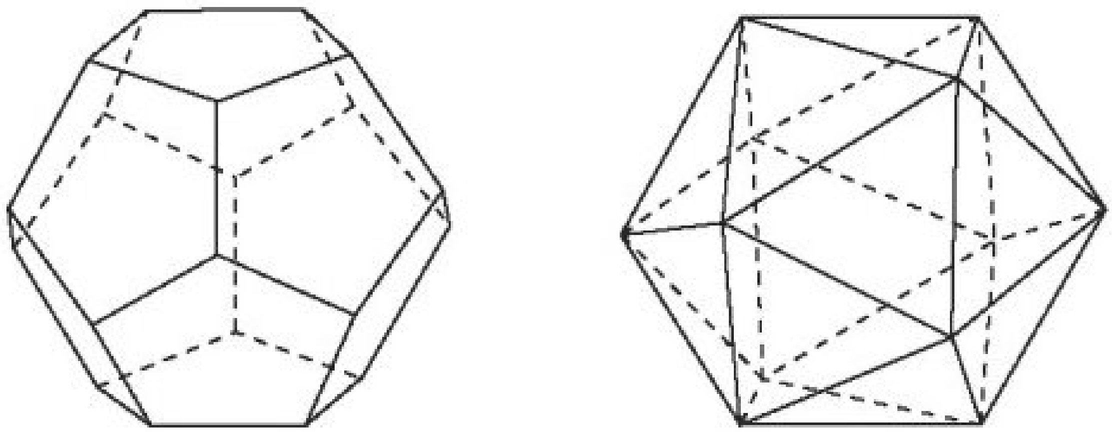

Figure 5: Dodecahedron and Icosahedron

The classification of all finite simple groups is one of the great achievements of XXth century mathematics.

The group of symmetries of the above Klein geometry is not finite since specifying one of these geometric transformations involves in fact choosing three continuous parameters. It falls under the theory of S. Lie which was in fact a direct continuation of the ideas formulated by Galois.

These ideas of Sophus Lie were reformulated in the “Erlangen program" of Félix Klein and successfully developed by Elie Cartan whose classification of Lie groups is another great success of XXth century mathematics. Through the work of Chevalley on algebraic groups the theory of Lie groups played a key role in the classification of finite simple groups.

-
-
-

Bibliography

[1] M.F. Atiyah, Global theory of elliptic operators, *Proc. Internat. Conf. on Functional Analysis and Related Topics* (Tokyo, 1969), University of Tokyo press, Tokyo (1970), 21-30. [This is a report on the generalization of the theory of Atiyah and Singer on the index of elliptic pseudo-differential operators.]

[2] L.G. Brown, R.G. Douglas and P.A. Fillmore, Extensions of C^*-algebras and K-homology, *Ann. of Math.* 2, **105** (1977), 265-324. [This is a fundamental paper on the so-called BDF theory.]

[3] P. Cartier, *A mad day's work: from Grothendieck to Connes and Kontsevich. The evolution of concepts of space and symmetry*, in "Les relations entre les mathématiques et la physique théorique", 23–42, Inst. Hautes Études Sci., Bures-sur-Yvette, 1998. (English translation in Bull. Amer. Math. Soc. (N.S.) 38 (2001), no. 4, 389–408). [This is an interesting story about three Fields medalist mathematicians and their mathematical research on space and symmetry, developed at Institute des Hautes Etades Scientifiques over something like 40 years period.]

[4] A. Connes, Une classification des facteurs de type III, *Ann. Sci. Ecole Norm. Sup*.,**6**, n. 4 (1973), 133-252. [This is the first detailed study of the structure of type III factors.]

[5] A. Connes Noncommutative geometry, Academic Press (1994). [This is a fundamental treatise of the non-commutative geometry, guiding readers the non-commutative geometry versions of four successively more refined kind of geometric structure: measure theory, topology(including algebraic geometry), differential topology and differential geometry, the last chapter containing interpretation on the Standard Model of particle physics.]

[6] A. Connes, Trace formula in Noncommutative Geometry and the zeros of the Riemann zeta function, *Selecta Mathematica* New Ser. 5 (1999) 29-106. [This is a proposal of a new approach to Riemann hypothesis as an application of non-commutative geometry.]

[7] A. Connes, M. Douglas, A. Schwarz, Noncommutative geometry and Matrix theory: compactification on tori, *J. High Energy Physics,* **2** (1998). [This is another attempt of applying the non-commutative geometry methods for the construction of a compactified non-commutative torus in connection with its physical significance.]

[8] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem. hep-th/9912092. Renormalization in quantum field theory and the Riemann-Hilbert problem II: The β function, diffeomorphisms and the renormalization group. hep-th/0003188. [These two papers give a self-contained treatment of the renormalization in quantum field theory in the framework of a general mathematical procedure of extraction of finite values based on the Riemann-Hilbert problem.]

[9] A. Connes and H. Moscovici, The local index formula in noncommutative geometry, GAFA, **5** (1995), 174-243. [This paper treats a local index formula in non-commutative geometry where the spaces can no longer be comprehended as point sets.]

[10] A. Connes, M. Marcolli, Renormalization and motivic Galois theory. (2004), Math NT/0409306. [This paper presents a non-commutative geometry treatment of an aspect of quantum field theory. Divergences in quantum field theory are organized in the structure of certain motivic Galois group, which is uniquely determined and universal with respect to the set of physical theory.]

[11] H. Freudenthal, *Lincos*, North Holland (1960). [This book proposes a language for communication with other intelligent creatures beyond the earth (if they exist), the central attempts being elimination of ambiguity and broadening of the list of discussible subjects (to the entire range of human experience).]

[12] A. Grothendieck, *Récoltes et semailles*, Université des Sciences et Techniques du Languedoc et Centre National de la Recherche Scientifique (1986). [Autobiography of the unique mathematician, Alexander Grothendieck.]

[13] G.G. Kasparov, The operator K-functor and extensions of C^* algebras, *Izv. Akad. Nauk SSSR, Ser. Mat.* **44** (1980), 571-636; *Math. USSR Izv.* **16** (1981), 513-572. [This paper presents the so-called K-K theory, generalizing the BDF theory (by Brown, Douglas and Fillmore) about the application of algebraic topology methods to operator algebras.]

[14] D. Kastler, Noncommutative geometry and fundamental physical interactions: The Lagrangian level, *Journal. Math. Phys.* (2000), 3867-3891. [This review is a nice introduction to the interplay of mathematics and theoretical physics (especially relevant to elementary particle theory), carrying the reader away on an enthusiastic trip inside the world of non-commutative geometry introduced and developed by A. Connes.]

[15] U. Kraehmer, Dirac Operators on Quantum Flag Manifolds. math.QA/0305071. [This paper presents an abstract framework for Dirac operators, yielding a Hilbert space realization of the covariant first-order differential calculi, where all differentials are bounded operators.]

[16] S. MacLane, I. Moerdijk, *Sheaves in Geometry and Logic*, Springer Verlag (1992). [This book is a very detailed introduction to topos theory, aimed at readers with relatively little background in category theory. Recognizing the difficulty of understanding abstract machinery when it is presented without adequate motivation, the authors introduce the main concept in a gradually increasing generality. The book, through its detailed discussions of points, through its many centrally important examples, and through its generally more friendly character will be a great help in the learning of the subject.]

[17] J. Martinet, J.P. Ramis, *Elementary acceleration and multisummability*, *I*, Ann. Inst. Henri Poincaré, Vol.54 (1991) 331–401. [This is a long survey paper devoted to a detailed examination of formal solutions of a class of first order differential operators and their representation as products involving so-called multisummable operators.]

[18] J. Milnor, D. Stasheff, Characteristic classes, *Ann. of Math. Stud.* , Princeton University Press, Princeton, N.J. (1974). [This is the fundamental textbook on characteristic classes.]

[19] M.A. Rieffel, Morita equivalence for C^*-algebras and W^*-algebras, *J. Pure Appl. Algebra* **5** (1974), 51-96. [This is a basic paper on Morita equivalence of C^*- and W^*-algebras which is widely used in theory of operator algebras.]

[20] B. Riemann, *Mathematical Werke*, Dover, New York (1953). [This is a collected mathematical works of Riemann, in which one finds his famous “Habilitationsvortrag”.]

[21] I.M. Singer, Future extensions of index theory and elliptic operators, *Ann. of Math. Studies* **70** (1971), 171-185. [This paper explains the widely known index theory for operators in a manifold.]

[22] M. Takesaki, *Theory of operator algebras I, II, III*, Springer-Verlag, Berlin, (vol.I, 2002; vol.II,III 2003). [This is one of standard textbooks on operator algebras, where the structure and classification theory of von Neumann algebras are treated in detail.]

Biographical Sketch

Alain Connes was born in Draguignan, a town near Cannes in the Provence- Alpes-Côte-d'Azur region of southeast France. He entered the École Normale Supérieure in Paris in 1966, graduating in 1970. After graduating, Connes became a researcher at the Centre National de la Recherche Scientifique. His thesis A classification of factors of type III was on operator algebras, in particular on von Neumann algebras, and the work was supervised by Jacque Dixmier. The thesis was presented to the École Normale Supérieure in 1973.

Connes spent the academic year 1974-75 at Queen's University in Ontario Canada. In 1976 he was appointed a lecturer at the University of Paris VI, then he was promoted to professor. He spent the year 1978-79 at the Institute for Advanced Study at Princeton. He left the University of Paris VI in 1980 but, the previous year, he had been appointed as professor at the Institut des Hautes Études Scientifiques at Bures-sur-Yvette. Connes still holds this professorship.

In 1981 Connes returned to the Centre National de la Recherche Scientifique, this time as its director of research. He held this post for eight years. Another position he was appointed to was professor at the Collège de France at Rue d'Ulm in Paris in 1984. Connes currently holds both the position in the Institut des Hautes Études Scientifiques and the one in the Collège de France.

Scientific Awards and Distinctions
Invited address, International Congress of Mathematicians, Vancouver (1974).
Invited plenary address, International Congress of Mathematicians, Helsinki (1978).
Invited address, International Congress of Mathematicians, Berkeley (1986).

Prix Aimée Berthé de l'Académie des Sciences (1975).
Prix Peccot-Vimont du Collège de France (1976).

Médaille d'argent du CNRS (1975).
Prix Ampère de l'Académie des Sciences (1980).
Fields Medal (1982).
Crafoord Prize (2001).

Doctor Honoris Causa, Queen's University, Kingston, Canada (1979).
Foreign Associate Member of the Academy of Sciences of Denmark (1980).
Member of l'Académie des Sciences de France (1983).
Foreign Honorary Member of the American Academy of Arts and Sciences (1990).
Foreign Associate Member of the Academy of Sciences of Norway (1993).
Foreign Fellow of the Royal Society of Canada (1996).
Doctor Honoris Causa, University of Rome Tor Vergata (1997).
Foreign Associate Member of the National Academy of Sciences USA (1997)